NONSTATIONARY (TRANSIENT) RADIATIVE-CONDUCTIVE HEAT TRANSFER IN A PLANE LAYER OF GRAY ABSORBING MEDIUM

A. L. Burka and N. A. Rubtsov

The kinetics of the heating of a plane layer of gray absorbing medium by radiative-conductive heat transfer are considered. The nonstationary energy equation is reduced to a nonlinear integral equation by means of a Green's function, and this is solved numerically by the Newton method. The results of the solution are presented in the form of the temperature fields in the layer for various values of the defining parameters (optical thickness, radiative-conductive heat-transfer criterion, heat-transfer criterion at the boundaries).

The heating of semi-transparent stationary media is effected by two time-varying mechanisms of heat transfer: molecular heat conduction and thermal radiation.

For a wide class of moderately rapid and rapid processes, radiative heat transfer depends only slightly on the time. Hence, in the radiative heat-transfer energy equation the term expressing the rate of change of the volume density of radiative energy is small, and may be neglected. Allowing for this limitation, the nonstationary energy equation of radiative-conductive heat transfer may be written as follows:

$$\operatorname{div}(\lambda \operatorname{grad} T) - \operatorname{div} \mathbf{E} = c\rho \frac{\partial T}{\partial t} \tag{1}$$

Here E is the radiation flux vector. The remaining notation is of the generally accepted type.

Problems in the form of the energy equation (1) arise from practical requirements of modern technical optics associated with the kinetics of heating (or cooling) semi-transparent heat-conducting materials (glass, crystals, etc.) by thermal radiation [1-3]. However, there are hardly any strict solutions of this kind of problem in existence.

In this paper we shall give a one-dimensional, numerical solution to a boundary problem based on the energy equation (1), in which div E has two representations: the integrated form, and the approximation of radiative heat conduction.

The integrated representation for div E [4] may be written:

$$\frac{d\Phi}{d\xi} = 2h\left[2\theta^4(\xi,\tau) - W_{1,h}(\xi) - \int_0^1 W_{z,h}(\xi,z)\,\theta^4(z,\tau)\,dz\right].$$

$$W_{1,h}(\xi) = \alpha\left[(a_1\theta_1^4 + 2a_2r_1\theta_2^4K_3(h))\,K_2(q) + (a_2\theta_2^4 + 2a_1r_2\theta_1^4K_3(h))\,K_2(h-q)\right] \tag{2}$$

$$W_{z,h}(\xi,z) = h\{K_1|q-p| + 2\alpha\left[r_1(K_2(p) + 2r_2K_3(h)\,K_2(h-p))\,K_2(q) + r_2(K_2(h-p) + 2r_1K_3(h)\,K_2(p)\,K_2(h-q))\right]\}$$

$$\alpha = \frac{1}{1-4r_1r_2K_3^2(h)}, \quad K_j(\xi) = \int_0^1 \nu^{j-2}\exp\frac{q}{\nu}\,d\nu, \quad p = hz, \; q = h\xi$$

In the approximation of radiative heat conduction, div E takes the form

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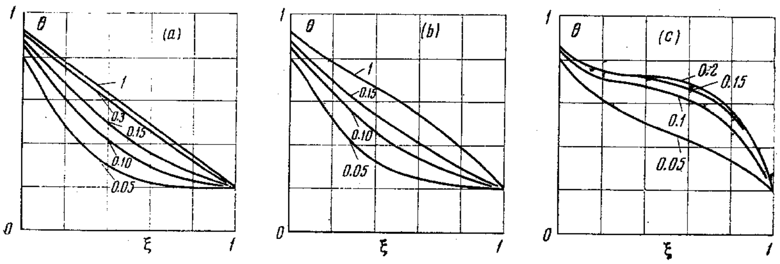

Fig. 1

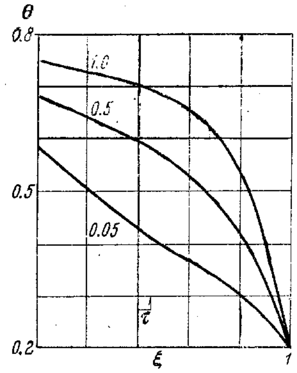

Fig. 2

$$\frac{d\Phi}{\partial\xi^2} = -\frac{d}{d\xi}\left(\frac{4}{3h}\theta^3\frac{d\theta}{d\xi}\right) \tag{3}$$

$$\left(\Phi = \frac{E}{\sigma_0 n^2 T_*^4},\ \theta = \frac{T}{T_*},\ h = \varkappa\delta,\ \theta_i = \left(\frac{E_i^*}{\sigma_0 n^2 T_*^4}\right)^{1/4}\right)$$

Here θ, T_*, σ_0, $\varkappa$, θ_i, E_i^*, a_i, r_i, k_j, h, ξ, and τ are, respectively, the unknown dimensionless temperature of the medium, the characteristic temperature of the medium, the Stefan–Boltzmann constant, the radiation absorption coefficient, the equivalent boundary temperatures, the hemispherical densities of the radiation falling on the corresponding surfaces of the layer, the absorbing and reflecting powers of the corresponding surfaces of the layer, the exponential integrals, the optical thickness of the layer, the dimensionless coordinates, and the time (i = 1, 2; j = 1, 2, 3). The physical and optical properties of the medium are assumed independent of temperature (n = 1).

The mathematical formulation of the boundary problem for the one-dimensional nonstationary energy equation may be written in the following dimensionless form in the case of a heat-conducting medium on allowing for (3):

$$\frac{\partial^2\theta}{\partial\xi^2} - S\frac{\partial\Phi}{\partial\xi} = \frac{\partial\theta}{\partial\tau},\qquad 0 \leqslant \xi \leqslant 1,\quad \tau > 0 \tag{4}$$

$$\alpha_1\frac{\partial\theta}{\partial\xi} - \beta_1(\theta - v_1) = 0,\qquad \xi = 0,\quad \tau > 0 \tag{5}$$

$$\alpha_2\frac{\partial\theta}{\partial\xi} + \beta_2(\theta - v_2) = 0,\qquad \xi = 1,\quad \tau > 0 \tag{6}$$

$$\theta(\xi, 0) = \theta_0,\qquad 0 \leqslant \xi \leqslant 1$$

$$\left(S = \frac{\sigma_0 T_*^3}{\lambda}\delta,\ \tau = \frac{at}{\delta^2},\ \frac{\beta_i}{\alpha_i} = \frac{\alpha_i\delta}{\lambda} = B\right) \tag{7}$$

Here v_i are the external temperatures of the medium, θ_0 is the initial temperature of the layer, S is the Stark criterion, α_i, β_i are certain positive constants not vanishing simultaneously, and B is the Biot criterion (i = 1, 2).

Let us consider the heating (cooling) of a flat layer of gray heat-conducting medium subjected to the external action of diffusely radiating and also convective heat flows. We shall assume that the rate of heating and the corresponding temperature drops are not so large as to necessitate allowing for the temperature dependence of the radiation absorption coefficients $\varkappa$ and the thermal conductivity λ.

As a defining temperature we take the temperature of the external, diffusely radiating source.

According to (2), $d\Phi/d\xi$ constitutes an integral equation nonlinear with respect to θ, so that Eq. (4) becomes a nonlinear integrodifferential equation. This prevents us from obtaining a solution to the boundary problem (4)-(7) in closed form. By using a Green's function we may reduce the boundary problem (4)-(7) to a functional equation (as we shall shortly show, it will be convenient to use iterative methods to solve this numerically)

$$\theta(\xi, \tau) = \frac{1}{\Delta}\{\beta_1 v_1[\alpha_2 \operatorname{ch}(1-\xi) + \beta_2 \operatorname{sh}(1-\xi)] + \beta_2 v_2[\alpha_1 \operatorname{ch}\xi + \beta_1 \operatorname{sh}\xi]\} + \int_0^1 G(\xi, z) F(\theta, z, \tau)\, dz \tag{8}$$

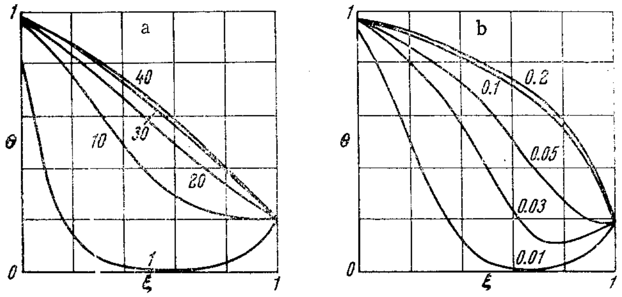

Fig. 3

Here

$$G(\xi, z) = \begin{cases} (\alpha_1 \operatorname{ch} z + \beta_1 \operatorname{sh} z)[\alpha_2 \operatorname{ch}(1-\xi) + \beta_2 \operatorname{sh}(1-\xi)]/\Delta, & z \leqslant \xi \\ (\alpha_1 \operatorname{ch} \xi + \beta_1 \operatorname{sh} \xi)[\alpha_2 \operatorname{ch}(1-z) + \beta_2 \operatorname{sh}(1-z)]/\Delta, & z > \xi \end{cases}$$

$$\Delta = -[(\alpha_1\beta_2 + \alpha_2\beta_1)\operatorname{ch}1 + (\alpha_1\beta_2 + \beta_1\beta_2)\operatorname{sh}1]$$

For purposes of numerical solution, Eq. (8) is converted into arithmetical form in the following sequence. The time derivative is approximated by a finite-difference operator. Then for each moment of time τ the integral is approximated by a Gauss quadrature formula. In this way Eq. (8) is reduced to a system of nonlinear algebraic equations. The Newton iteration method may be applied to the numerical solution of this kind of equation [5].

The results of the solution presented in Figs. 1-3 reflect the heating kinetics of a gray heat-conducting medium when one of the surfaces of the layer ($\xi = 0$) is heated by a convective flow with the temperature of the external medium $v_1 = 1$, $B_1 = 10$, $\theta_1 = 1$ and the other surface of the layer ($\xi = 1$) is held at a temperature $v_2 = 0.2$ ($a_1 = 0.2$, $a_2 = 0.8$, $r_1 = 0.8$, $r_2 = 0.2$). In Fig. 1a the temperature distribution in the layer is given for $\theta_0 = 0.2$, $S = 0.25$, $h = 1$ and various times τ.

It is a characteristic feature that, on passing to the steady state mode ($\tau = 1$), the temperature curve has a linear character, as in the case of pure heat conduction. This is due to the fact that for $S = 0.25$ the proportion of radiative heat transfer is not particularly great.

Figure 1b shows the temperature field under the same conditions as in Fig. 1a, except that $S = 2.5$. We see from the figure that, on passing to the steady state mode, the temperature curves change their character and become convex. This indicates the greater intensity of heating attributable to the radiative component of the total heat transfer.

In Fig. 1c, which relates to $S = 25.0$ ($\theta_0 = 0$, $v_2 = 0$) we have a more obvious deformation of the temperature profiles than before. This is due to the predominant role of radiation as opposed to molecular heat transfer. As the contribution of the radiative component to the total heat transfer increases, the time required for the system to pass into the steady state mode diminishes. This is particularly apparent in the region of the boundary adjacent to the heat source. On reducing the intensity of heat access, not only the general level of temperature but also the rate of heating diminish (Fig. 2, in which $B_1 = 5$ instead of the previous $B_1 = 10$). We must say a few words on the convergence of the iteration process used in the solution of Eq. (8). The process converges rapidly for $0 \le h \le h_{max}$, $S \le 1$. Starting from a certain value of $h \ge h_{max}$ and $S > 1$, the convergence of the process worsens sharply. Starting from a certain instant of time τ, the temperature profiles become unstable. In order to remove these harmful effects, the number of Gauss points has to be increased, and this means increasing the order of the system of equations and hence greatly increasing computer time.

If the optical thickness of the layer is sufficiently great, the energy equation (1) may be written in the following way, allowing for the approximate representation (3) for the radiation flux vector:

$$\frac{\partial^2\theta}{\partial\xi^2} + \frac{\partial}{\partial\xi}\left(\frac{16}{3h} S\theta^3 \frac{\partial\theta}{\partial\xi}\right) = \frac{\partial\theta}{\partial\tau}, \quad 0 \leqslant \xi \leqslant 1, \quad \tau > 0 \tag{9}$$

Of practical interest is the solution of Eq. (9) with the following boundary conditions:

$$\frac{\partial\theta}{\partial\xi} = S[\theta^4(\xi, \tau) - 1], \qquad \xi = 0 \tag{10}$$

$$\theta(\xi,\tau)=\theta_1,\ \xi=1;\quad \theta(\xi,\tau)=\theta_0,\quad \tau=0 \tag{11}$$

The boundary problem (10), (11) may be reduced by means of a Green's function to the functional equation

$$F(\theta)=0 \tag{12}$$

Here

$$F(\theta)=\left[\theta_1-\theta(\xi,\tau)+\int_0^1 G(\xi,z)\frac{\partial\theta}{\partial\tau}dz\right]+S\left\{(1-\xi)\left[1-\theta^4(0,\tau)\right]\right.$$
$$\left.\times\left[1+\frac{16}{3h}S\theta^3(0,\tau)\right]+\frac{4}{3h}\left[\theta_1^4-\theta^4(\xi,\tau)\right]\right\},\quad G(\xi,z)=\begin{cases}(\xi-1), & z\leqslant\xi\\(z-1), & z\geqslant\xi\end{cases}$$

The functional equation (12) contains two unknown functions $\theta(\xi,\tau)$ and $\theta(0,\tau)$. Putting $\xi=0$, we derive yet another equation from (12) to close the system. Thus the original boundary problem (10), (11) reduces to a system of two functional equations

$$F[\theta(\xi,\tau),\ \theta(0,\tau)]=0,\quad F_1[\theta(\xi,\tau),\ \theta(0,\tau)]=0 \tag{13}$$

As before, we carry out an arithmetic conversion and reduce (13) to a system of (m + 1) nonlinear algebraic equations for which a numerical solution may be achieved by the Newton method of iteration.

Some results obtained by a numerical solution of Eq. (13) for the conditions $\theta_0=0$, $\theta_1=1$, S = 10 and various values of h are presented in Fig. 3.

Figure 3a and 3b shows the results of the calculation for h = 40 and h = 4. The role of the optical thickness in creating the temperature field for a fixed parameter S (characterizing the radiation-conduction relationship in the total heat flow) appears very clearly in these figures.

This is reflected in the character of the temperature profiles throughout the whole layer, including the extremely characteristic temperature jumps in time at the boundary. We see that on increasing h the rate of heating of the whole layer diminishes considerably.

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